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A note on the instability of columnar vortices

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The inviscid compressible instability of columnar-vortex flows to three-dimensional perturbations with large wavenumbers is considered. The sufficient conditions for instability obtained are compared with recently published results for incompressible fluids.

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1. Introduction

The problem of stability of helical flows has been studied in a number of papers over the years. Recently, Leibovich & Stewartson (1983) studied the problem for an unbounded, homogeneous and inviscid fluid subject to infinitesimal three-dimensional disturbances. By using an asymptotic analysis for large azimuthal wavenumbers, they were able to obtain a sufficient condition for instability of columnar vortices:

$$\frac{V}{r}\left(V'-\frac{V}{r}\right)\left\{V'^2+W'^2-\left(\frac{V}{r}\right)^2\right\}<0, \tag{1.1}$$

where r denotes the distance from the axis of symmetry, $W(r)$ the axial velocity and $V(r)$ the azimuthal velocity of the basic flow. By comparison with numerical solutions for a specific family of flows, Leibovich & Stewartson also found that the asymptotic theory predicts the most-unstable wave with reasonable accuracy for values of the azimuthal wavenumber n as low as 3, and that it improves rapidly in accuracy as n increases.

The main purpose of the work by Leibovich & Stewartson (1983) was to study the role played by hydrodynamic instabilities in phenomena such as vortex breakdown that are known to occur for instance in trailing line vortices downstream of the wingtips of aircraft. The speeds involved in these phenomena are usually low compared with the speed of sound, and almost all systematic studies available so far seem to have been carried out at low speeds. Leibovich & Stewartson therefore restricted their study to an incompressible homogeneous fluid. For large aircraft, however, the trailing vortices are so strong that they may cause serious hazards for a following smaller aircraft (Chigier 1974). In such cases the speeds involved and the resulting density stratification of the fluid in the vortices must be significant; it is therefore not clear *a priori* that the effects of compressibility and stratification are always negligible in the associated stability problem. Thus it should be of interest to extend the results obtained by Leibovich & Stewartson to compressible fluids. This extension has, however, already been partly done by Eckhoff & Storesletten (1978) in a somewhat more general setting where a special family of external forces as well as possible tubular boundaries are taken into account.

The asymptotic analysis carried out by Leibovich & Stewartson for the normal modes is of the WKB type. The generalized progressing-wave expansion method used by Eckhoff & Storesletten is different, but is also a generalization of the WKB method.

It is an asymptotic method valid for large wavenumbers, and it has been proved by Eckhoff (1981) that it can be used to establish sufficient conditions for instability as anticipated by Eckhoff & Storesletten. Since the WKB method is the starting point in either of the two reported ways of approaching the stability problem for columnar vortices, it must be anticipated that the results are directly comparable. The purpose of this note is to show the expected conformity of the results and to make some remarks on the differences found for the two models and the two approaches. In particular we shall show that the extension of the condition (1.1) to compressible fluids can easily be extracted from the results obtained by Eckhoff & Storesletten.

2. Discussion of stability in the compressible case

In addition to the notation introduced in §1, we shall let $\rho_0(r)$ denote the density, $p_0(r)$ the pressure and $c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ the local sound speed in the basic flow, where γ is a constant. Restricting ourselves to the case with no external forces and assuming that $V(r) \neq 0$ everywhere, the theorem proved by Eckhoff & Storesletten (1978) then states that a necessary condition for stability of the basic flow is that

$$\frac{V^2}{r}\left\{\frac{\rho_0'}{\rho_0}-\frac{V^2}{c_0^2 r}\right\} > -\frac{V}{r}\left(\frac{V}{r}+V'\right)+\left\{\left(\frac{V}{r}\right)^2\left[\left(\frac{V}{r}+V'\right)^2+W'^2\right]\right\}^{\frac{1}{2}} \tag{2.1}$$

almost everywhere in the fluid.

Looking at the proof of this result in §3 of Eckhoff & Storesletten (1978), one will find that most of the instabilities detected when (2.1) is violated are perturbations with an algebraic growth in t. Exponential growth was only found when the system of transport equations was autonomous, which occurred when

$$\frac{\xi^2}{r}\left(\frac{V}{r}-V'\right)-\xi^3 W' = 0, \tag{2.2}$$

where ξ^2 and ξ^3 are analogous to local azimuthal and axial wavenumbers respectively. When (2.2) is satisfied, the *local* fluid-oscillation frequencies were found to be (for the full frequencies see §4)

$$\pm\frac{D}{\{(\xi^1)^2+(r^{-1}\xi^2)^2+(\xi^3)^2\}^{\frac{1}{2}}}, \tag{2.3}$$

where ξ^1 is analogous to the local radial wavenumber, and D is given by

$$D^2 = \left(\frac{\xi^2}{r}\right)^2\frac{V^2}{r}\left(\frac{\rho_0'}{\rho_0}-\frac{V^2}{c_0^2 r}\right)-\frac{2\xi^2\xi^3 VW'}{r^2}+(\xi^3)^2\left[\frac{V^2}{r}\left(\frac{\rho_0'}{\rho_0}-\frac{V^2}{c_0^2 r}\right)+\frac{2V}{r}\left(\frac{V}{r}+V'\right)\right]. \tag{2.4}$$

Thus we see that a sufficient condition for exponential instabilities in the compressible case is that somewhere in the fluid $D^2 < 0$ when (2.2) holds. Introducing (2.2) into (2.4), this sufficient condition for exponential instability is easily found to be

$$\frac{V^2}{r}\left\{\left(V'-\frac{V}{r}\right)^2+W'^2\right\}\left(\frac{\rho_0'}{\rho_0}-\frac{V^2}{c_0^2 r}\right)+\frac{2V}{r}\left(V'-\frac{V}{r}\right)\left\{V'^2+W'^2-\left(\frac{V}{r}\right)^2\right\} < 0. \tag{2.5}$$

Let us denote the quantity on the left-hand side in (2.5) by $-\mu^2$; the corresponding maximum growth rate is then obtained by introducing (2.2) and (2.4) into (2.3) and setting $\xi^1 = 0$. This immediately gives

$$\mu\left\{\left(V'-\frac{V}{r}\right)^2+W'^2\right\}^{-\frac{1}{2}} \tag{2.6}$$

for the maximum local growth rate of the considered exponentially unstable perturbations.

3. The incompressible limit

If we restrict our attention to exponentially growing disturbances, as was done by Leibovich & Stewartson (1983), our sufficient condition for instability (2.5) is seen to reduce to the condition (1.1) in the incompressible (and homogeneous) limit $\rho_0' \rightarrow 0$, $c_0 \rightarrow \infty$. Also, the relation (2.2) for the wavenumbers and our estimate (2.6) for the maximum growth rate are seen to match the ones found by Leibovich & Stewartson (1983, equation (5.6) and (5.8) respectively) in this limit.

It is not as simple to consider the incompressible limit for the algebraically growing disturbances considered by Eckhoff & Storesletten (1978). In fact, a naive application of the results obtained in that paper may indicate that the basic flow is always unstable, since the inequality (2.1) can never be fulfilled in that limit. A more detailed analysis is needed, however, to discuss the stability behaviour properly, since the growth rates for the growing perturbations may tend to zero in the limit $\rho_0' \rightarrow 0$, $c_0 \rightarrow \infty$. This limit should therefore be taken in the system of transport equations (Eckhoff & Storesletten 1978, equations (2.15)–(2.17)), and the stability properties of the resulting system then discussed. By an analysis completely analogous to that in Eckhoff & Storesletten (1978, §3), it is found that, when (2.2) is not satisfied, the system of transport equations tends asymptotically as $\tau = \ln t \rightarrow +\infty$ to the following system:

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\tau} = \boldsymbol{B}_\mathrm{i}\boldsymbol{\sigma}, \tag{3.1}$$

where

$$\boldsymbol{B}_\mathrm{i} = \frac{1}{e}\begin{pmatrix} -\dfrac{\xi^2}{r}\left(\dfrac{V}{r} - V'\right) & 0 & -\xi^3\left(\dfrac{V}{r} + V'\right) \\ 0 & 0 & 0 \\ -\dfrac{\xi^2 W'}{r} + \dfrac{2\xi^3 V}{r} & 0 & \xi^3 W' \end{pmatrix}, \tag{3.2}$$

$$e = \frac{\xi^2}{r}\left(\frac{V}{r} - V'\right) - \xi^3 W'. \tag{3.3}$$

The matrix $\boldsymbol{B}_\mathrm{i}$ is the incompressible limit of the corresponding matrix $\boldsymbol{B}_0$ in Eckhoff & Storesletten (1978, equation (3.4)). Hence the eigenvalues of $\boldsymbol{B}_\mathrm{i}$ are

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{2} + \left\{\frac{1}{4} - \left(\frac{D_\mathrm{i}}{e}\right)^2\right\}^{\frac{1}{2}}, \quad \lambda_3 = -\frac{1}{2} - \left\{\frac{1}{4} - \left(\frac{D_\mathrm{i}}{e}\right)^2\right\}^{\frac{1}{2}}, \tag{3.4}$$

where D_i is the incompressible limit of D given by (2.4), i.e.

$$D_i^2 = -\frac{2\xi^2\xi^3 V W'}{r^2} + \frac{2(\xi^3)^2 V}{r}\left(\frac{V}{r} + V'\right). \tag{3.5}$$

Asymptotically as $t \rightarrow \infty$ the system of transport equations therefore has solutions that are proportional to each of the three factors t^{λ_j}, $j = 1, 2, 3$. Thus a necessary condition for stability is seen to be that $D_\mathrm{i}^2 \geqslant 0$ for every possible choice of wavenumbers ξ^2, ξ^3. From (3.5) this is easily seen to imply the following necessary conditions for stability of the basic flow when $V \neq 0$ everywhere:

$$W' = 0, \tag{3.6}$$

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(rV)^2 = \frac{2V}{r}\left(\frac{V}{r} + V'\right) > 0. \tag{3.7}$$

If either (3.6) or (3.7) is violated somewhere in the fluid, an algebraically growing disturbance exists in the incompressible limit. If (3.6) and (3.7) are satisfied, the trivial solution of the transport equations is stable in this limit.

Condition (3.7) is exactly the classical Rayleigh condition, while (3.6) means that spiral flows are always unstable in the incompressible limit. However, it does not necessarily follow from this that spiral flows are always unstable for the incompressible-fluid model. In fact, from a mathematical point of view that model is an extremely singular limit of the compressible-fluid model. In particular we see that a compressible fluid allows for a larger class of perturbations than an incompressible fluid, since no perturbations of the density are allowed in the latter. Thus it should not be surprising if some instabilities found for compressible fluids do not have counterparts in the incompressible case, but I do not know of any discussion on this problem in the literature so far. Presumably it is always possible to show that exponential instabilities for compressible fluids have counterparts for incompressible fluids, so the discrepancies are most likely limited to the marginal cases where only algebraically growing disturbances can be found. The method of analysis used by Leibovich & Stewartson (1983) does not cover such marginal instabilities; it is therefore an open question whether they exist for an inviscid incompressible fluid.

4. Some remarks

Since trailing vortices are formed continuously by the wings of an aircraft, it may seem a reasonable approximation to assume that the flow field is homentropic, i.e. that the fluid has uniform entropy, at least in the neighbourhood of the wings. Since obviously viscosity will have an effect in the core regions of the vortices, the flow field will certainly change downstream. However, I do not know of any systematic study where the developments of the density and the pressure structure in a trailing vortex are studied.

If we assume a homentropic flow, we find that

$$\frac{\rho_0'}{\rho_0} - \frac{V^2}{c_0^2 r} = 0. \tag{4.1}$$

Hence the sufficient condition for exponential instability (2.5) reduces for such flows to the condition (1.1) found for incompressible flows, no matter how large the stratification is. Equation (4.1) simply means that the effect of compressibility and the effect of stratification exactly cancel each other in the problem considered.

If (4.1) is satisfied, however, the condition (2.1) can never be fulfilled for vortex flows with $W' \neq 0$. Thus the linear theory of inviscid compressible fluids predicts algebraically growing disturbances in this case. Looking at the structure of the linear perturbation equations governing these instabilities, we will find that to leading order only the velocity components in the azimuthal and axial directions are affected. In this approximation the perturbations of the other field variables are either stable or growing at a slower rate. If the algebraic instabilities play a role in the development of the vortex structure downstream, we may therefore expect a redistribution in the velocity field in the axial and azimuthal directions. Such redistributions in the velocity field are observed in trailing vortices, but it is an open question whether the algebraic instabilities detected play any role in this development, or whether this development is totally dominated by the viscosity in the core region of the vortex. Presumably the development of the vortex structure downstream will in any case lead to growth of the left-hand side in (4.1). We may therefore expect the terms on the far-left side of (2.5) to be positive far downstream of the wingtip, thus having a stabilizing influence on the exponentially growing disturbances. For a trailing vortex, however, the contributions due to stratification and compressibility are presumably very small compared with the other terms in (2.5); hence they can probably be neglected in most cases. If on the other hand we consider flows encountered in a gas centrifuge for instance, the terms due to stratification and compressibility are definitely not negligible *a priori*.

Discussion of the effects of compressibility and stratification also parallels the above one for the frequencies of the modes when the sufficient conditions for instability (1.1) and (2.5) are not satisfied. In the autonomous case where (2.2) is satisfied, we see from Eckhoff & Storesletten (1978, equations (2.10*a*), (2.12*d–f*), (A 2), (A 7)) that the frequency due to the principal phase function $\Phi(r, \phi, z, t)$ is

$$-\omega\Phi_t = \omega\left(\frac{\xi^2 V}{r}+\xi^3 W\right). \tag{4.2}$$

Here ω is an expansion parameter, and with the notation of Leibovich & Stewartson (1983) we have $n = -\omega\xi^2$ and $\alpha = \omega\xi^3$. Since the amplitudes in the generalized progressing-wave expansions oscillate with the frequencies (2.3), the full frequencies in the leading terms of the expansions are the sum of (4.2) and (2.8), in correspondence with the results found by Leibovich and Stewartson (1983). For a special family of flows Leibovich & Stewartson also calculated corrections to the leading-order frequencies in the incompressible case. Presumably it is also possible to obtain corrections in the compressible case by expansions similar to the uniform expansions constructed in Eckhoff (1982). Those expansions have so far only been worked out for simple roots in the characteristic equation, however, while the inertial waves are determined by a triple root.

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